

Duality in linearized gravity and holography

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Abstract

We consider spherical gravitational perturbations of AdS_4 space-time satisfying general boundary conditions at spatial infinity. Using the holographic renormalization method, we compute the energy-momentum tensor and show that it can always be cast in the form of Cotton tensor for a dual boundary metric. In particular, axial and polar perturbations obeying the same boundary conditions for the effective Schrödinger wave-functions exhibit an energy-momentum/Cotton tensor duality at conformal infinity. We demonstrate explicitly that this is holographic manifestation of the electric/magnetic duality of linearized gravity in the bulk, which simply exchanges axial with polar perturbations of AdS_4 space-time. We note on the side that this particular realization of gravitational duality is also valid for perturbations near flat and dS_4 space-time, depending on the value of cosmological constant.

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1 Introduction

AdS/CFT correspondence, [1], [2], [3], provides a very broad framework to study the relation between gravitational theories in $d + 1$ space-time dimensions and conformal field theories in d dimensions. Most of the work that has been carried out so far focused on $d = 4$ in connection with supersymmetric Yang-Mills theories living on the boundary of AdS_5 space-time. More recently, there has been increased interest in aspects of AdS_4/CFT_3 correspondence, whose field theory side is much less understood. The purpose of the present work is to exploit some special properties of four-dimensional gravity and obtain their holographic manifestation on the three-dimensional boundary. Hopefully, this development will help to shed light into the nature of the boundary field theory and its duality symmetries in future studies.

We will focus on the simplest case of AdS_4 space-time (without black holes) and derive some properties of the energy-momentum tensor for general gravitational perturbations satisfying arbitrary boundary conditions, since they are legitimate to consider by the analysis of ref. [4]. It is useful, in this context, to split the perturbations into two distinct classes and use the linearized Einstein equations to establish a duality among them. As will be explained in the sequel, this provides a concrete realization of the electric/magnetic duality in linearized gravity, which is rather specific to four space-time dimensions, and it manifests as energy-momentum/Cotton tensor duality on the three-dimensional conformal boundary. Setting the boundary free in AdS_4/CFT_3 correspondence, as in ref. [5], is the necessary ingredient in order to establish these general results. Actually, the existence of a dual graviton correspondence was anticipated before by some general arguments in the AdS_4/CFT_3 case, [6], but a concrete realization of it

was still lacking. Our work fills up this gap and paves the way for a deeper understanding of the energy-momentum tensor/Cotton tensor duality for AdS_4 black holes that was observed recently, [7].

The realization of the gravitational electric/magnetic duality that will be obtained along the way has more general value, beyond holography, since it is totally independent of the size and sign of the cosmological constant. In fact, it provides for the first time an explicit solution of the non-local transformation law that connects general metric perturbations around $(A)dS_4$ and/or Minkowski space-time, depending on the value of Λ , which are dual to each other. Our analysis also suggests possible generalizations of the electric/magnetic duality of linearized gravity around non-trivial backgrounds, such as the Schwarzschild solution, which remain to be worked out in detail and applied to the general theory of gravitational quasi-normal modes with or without cosmological constant. Thus, the results we present here can be regarded as the simplest instance of a more general program aiming at the origin of gravitational duality and its holographic implications, when $\Lambda < 0$.

The material of this paper is organized as follows: In section 2, we consider the most general spherical perturbations of AdS_4 space-time and derive the equations governing the metric components in the linear approximation. In section 3, we study the effective Schrödinger equation governing the perturbations and determine the spectrum of oscillations under general boundary conditions at $r = \infty$. In section 4, we use the holographic renormalization method to compute the energy-momentum tensor at the conformal boundary of space-time for general perturbations satisfying arbitrary boundary conditions. In section 5, we show that the energy-momentum tensor can be cast in the form of Cotton tensor for a dual boundary metric, which arises by exchanging axial with polar perturbations satisfying the same boundary conditions. In section 6, we show that the holographic dual graviton correspondence in AdS_4 space-time is manifestation of the electric/magnetic duality of linearized gravity in the bulk that simply exchanges axial with polar perturbations. Finally, in section 7, we present the conclusions and summarize some open questions and directions for future work.

2 Spherical perturbations of AdS_4 space-time

Einstein equations in four space-time dimensions with negative cosmological constant, $R_{\mu\nu} = \Lambda g_{\mu\nu}$, admit AdS_4 space-time as solution. The metric is globally defined using spherical coordinates

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \quad (2.1)$$

with

$$f(r) = 1 - \frac{\Lambda}{3}r^2. \quad (2.2)$$

We shall consider perturbations of the metric around the spherically symmetric static configuration satisfying the linearized Einstein equations

$$\delta R_{\mu\nu} = \Lambda \delta g_{\mu\nu} . \quad (2.3)$$

In four space-time dimensions, there are two complementary classes of metric perturbations with opposite parity called axial and polar. In both cases, the equations reduce to an effective Schrödinger problem in the radial direction and all components of the metric are expressed in terms of the effective wave-function. The energy levels determine the allowed frequencies of oscillation around the static solution and their values depend on the boundary conditions imposed at spatial infinity. The result can be regarded as specialization of the general theory of quasi-normal modes of AdS_4 black holes in the limit of vanishing mass, $m = 0$, so that global AdS_4 space-time is considered instead. Naturally, many simplifications occur in this case and there are additional features not to be found when $m \neq 0$. The reader may consult refs. [8], [9], [10] for the general theory of quasi-normal modes and refs. [11], [12], [13] for the generalization to AdS_4 black holes.

Let us define for the purposes of the present work the tortoise radial coordinate r_* ,

$$dr_* = \frac{dr}{f(r)} , \quad (2.4)$$

so that upon integration we obtain

$$\tan \left(\sqrt{-\frac{\Lambda}{3}} r_* \right) = \sqrt{-\frac{\Lambda}{3}} r . \quad (2.5)$$

Let us also define for convenience the angular variable

$$x = \sqrt{-\frac{\Lambda}{3}} r_* , \quad (2.6)$$

which assumes all values from 0 to $\pi/2$ as r varies from the origin $r = 0$ to spatial infinity $r = \infty$. Then, the effective Schrödinger problem governing the perturbations of AdS_4 space-time assumes the following form,

$$\left(-\frac{d^2}{dx^2} + V(x) \right) \Psi(x) = \Omega^2 \Psi(x) , \quad (2.7)$$

for appropriately chosen potential $V(x)$.

Next, we examine separately the two distinct classes of metric perturbations, $g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}$, and tabulate them by matrices labeled by (t, r, θ, ϕ) . Without great loss of generality, and to simplify the presentation, we only consider axially symmetric perturbations parametrized by the Legendre polynomials $P_l(\cos\theta)$; of course, more general perturbations are expressed in terms of spherical harmonics $Y_l^m(\theta, \phi)$, but the main results remain essentially the same.

(i). **Axial perturbations** : The first class of metric perturbations of AdS_4 space-time assume the following form

$$\delta g_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0(r) & h_1(r) & 0 & 0 \end{pmatrix} e^{-i\omega t} \sin\theta \partial_\theta P_l(\cos\theta) , \quad (2.8)$$

The allowed frequencies of oscillation ω around the equilibrium configuration will be discussed in the next section. Axial perturbations correspond to the so called vector sector or shear channel in the dictionary of AdS/CFT correspondence.

It turns out, without giving all details, that the coefficients of the metric perturbations are determined in terms of a single function $\Psi_{\text{RW}}(x)$ that satisfies an effective Schrödinger equation (2.7) in the variable x with potential

$$V_{\text{RW}}(x) = \frac{l(l+1)}{\sin^2 x} \quad (2.9)$$

and

$$\Omega = \sqrt{-\frac{3}{\Lambda}} \omega . \quad (2.10)$$

In particular, Einstein equations amount to the following relations for the metric functions,

$$h_0(x) = \frac{i}{\omega} \frac{d}{dx} (\tan x \Psi_{\text{RW}}(x)) , \quad (2.11)$$

$$h_1(x) = \sqrt{-\frac{3}{\Lambda}} \sin x \cos x \Psi_{\text{RW}}(x) . \quad (2.12)$$

Here, we refer to the quantities of the problem as Regge-Wheeler potential and wave-functions, using the analogy with the axial perturbations of four-dimensional black holes, [8]. Different boundary conditions at spatial infinity, $r = \infty$, will be encoded into the behavior of the wave-function $\Psi_{\text{RW}}(x)$ at $x = \pi/2$ and specify the solution.

(ii). **Polar perturbations** : This is a complementary class of metric perturbations parametrized by four arbitrary radial functions of the general form

$$\delta g_{\mu\nu} = \begin{pmatrix} f(r)H_0(r) & H_1(r) & 0 & 0 \\ H_1(r) & H_2(r)/f(r) & 0 & 0 \\ 0 & 0 & r^2 K(r) & 0 \\ 0 & 0 & 0 & r^2 K(r) \sin^2\theta \end{pmatrix} e^{-i\omega t} P_l(\cos\theta) \quad (2.13)$$

They correspond to the so called scalar sector or sound channel in the dictionary of AdS/CFT correspondence. The frequencies of oscillation ω are in general different for the axial and polar perturbations, depending on the boundary conditions imposed on each sector.

It turns out, as before, that the coefficients of the metric perturbations are determined in terms of a single function $\Psi_Z(x)$ that satisfies an effective Schrödinger equation (2.7) in the variable x with potential

$$V_Z(x) = \frac{l(l+1)}{\sin^2 x} \quad (2.14)$$

that is identical to $V_{RW}(x)$. Also, Ω is expressed in terms of the allowed frequencies ω by equation (2.10). The linearized Einstein equations are satisfied provided that

$$H_0(r) = H_2(r) . \quad (2.15)$$

Furthermore, by the same token, the remaining coefficients of the metric perturbation are expressed in terms $\Psi_Z(x)$ as follows,

$$H_0(x) = \sqrt{-\frac{\Lambda}{3}} \left(\frac{l(l+1)}{2} \cot x + \frac{3\omega^2}{\Lambda} \sin x \cos x + \cos^2 x \frac{d}{dx} \right) \Psi_Z(x) , \quad (2.16)$$

$$H_1(x) = -i\omega \cos x \frac{d}{dx} (\sin x \Psi_Z(x)) , \quad (2.17)$$

$$K(x) = \sqrt{-\frac{\Lambda}{3}} \left(\frac{l(l+1)}{2} \cot x + \frac{d}{dx} \right) \Psi_Z(x) . \quad (2.18)$$

Here, we refer to the quantities of the problem as Zerilli potential and wave-functions, using the analogy with the polar perturbations of four-dimensional black holes, [9]. However, unlike the case of black holes, which exhibit different effective potentials for the axial and polar perturbations, the perturbations of AdS_4 space-time are governed by the same potential. As before, different boundary conditions at spatial infinity, $r = \infty$, will be encoded into the behavior of the wave-function $\Psi_Z(x)$ at $x = \pi/2$.

3 Boundary conditions and spectrum

For both axial and polar perturbations of AdS_4 space-time one is led to consider the effective Schrödinger problem

$$\left(-\frac{d^2}{dx^2} + \frac{l(l+1)}{\sin^2 x} \right) \Psi(x) = \Omega^2 \Psi(x) , \quad (3.1)$$

where the frequencies of perturbation are given by

$$\Omega = \sqrt{-\frac{3}{\Lambda}} \omega . \quad (3.2)$$

This problem can be transformed into a hypergeometric differential equation, using the change of variables (see, for instance, refs. [4], [12])

$$z = \sin^2 x, \quad \Psi(x) = \cos x \sin^{l+1} x Y(z), \quad (3.3)$$

namely

$$z(1-z) \frac{d^2 Y}{dz^2} + [c - (a+b+1)z] \frac{dY}{dz} - ab Y = 0 \quad (3.4)$$

with coefficients

$$a = \frac{1}{2}(l+2+\Omega), \quad b = \frac{1}{2}(l+2-\Omega) \quad (3.5)$$

and

$$c = l + \frac{3}{2}. \quad (3.6)$$

Then, the solution of the effective Schrödinger equation is expressed in terms of hypergeometric functions (in standard notation) as follows,

$$\Psi(x) = \cos x \sin^{l+1} x F(a, b; c; \sin^2 x), \quad (3.7)$$

up to an overall numerical factor, so that $\Psi(0) = 0$ at the origin $r = 0$. It provides the normalizable solution of the equation, whereas the other mathematical solution

$$\Psi(x) = \frac{\cos x}{\sin^l x} F(a+1-c, b+1-c; 2-c; \sin^2 x) \quad (3.8)$$

is not normalizable and blows up at $r = 0$. The two solutions have the familiar x^{l+1} and x^{-l} behavior, respectively, near the origin. Here, we will only consider the first one for deriving the energy-momentum tensor of perturbed AdS_4 space-time; of course, both solutions should be taken into account to derive the two-point functions of the energy-momentum tensor, but this computation is beyond the purpose of the present work.

The behavior of the normalizable solution at spatial infinity follows by rewriting $\Gamma(a, b; c; z)$ in terms of the complementary argument $1-z = \cos^2 x$. Using standard identities of hypergeometric functions, it follows that

$$\begin{aligned} \Psi(x) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \cos x \sin^{l+1} x F(a, b; a+b+1-c; \cos^2 x) + \\ &\quad \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \sin^{l+1} x F(c-a, c-b; 1+c-a-b; \cos^2 x). \end{aligned} \quad (3.9)$$

The wave-function can be expanded in powers of $1/r$, as

$$\Psi(r) = I_0 + \frac{I_1}{r} + \frac{I_2}{r^2} + \frac{I_3}{r^3} + \frac{I_4}{r^4} + \dots, \quad (3.10)$$

and the first two coefficient turn out to be

$$I_0 = \Gamma^{-1} \left(\frac{1}{2}(l+2+\Omega) \right) \Gamma^{-1} \left(\frac{1}{2}(l+2-\Omega) \right) \quad (3.11)$$

$$I_1 = -2\sqrt{-\frac{3}{\Lambda}} \Gamma^{-1} \left(\frac{1}{2}(l+1+\Omega) \right) \Gamma^{-1} \left(\frac{1}{2}(l+1-\Omega) \right), \quad (3.12)$$

up to an overall (irrelevant) numerical factor.

The remaining coefficients of the asymptotic expansion of Ψ in powers of $1/r$ can be easily determined as

$$I_2 = -\frac{3I_0}{2\Lambda} \left(l(l+1) + \frac{3\omega^2}{\Lambda} \right), \quad (3.13)$$

$$I_3 = -\frac{I_1}{2\Lambda} \left((l-1)(l+2) + \frac{3\omega^2}{\Lambda} \right), \quad (3.14)$$

$$I_4 = \frac{3I_0}{8\Lambda^2} \left[\left(l(l+1) + \frac{3\omega^2}{\Lambda} \right)^2 - 6 \left(l(l+1) + \frac{4\omega^2}{\Lambda} \right) \right] \quad (3.15)$$

and so on. They will be need later (up to the order shown above) for the computation of the energy-momentum tensor.

The boundary conditions at spatial infinity $r = \infty$ are solely expressed in terms of I_0 and I_1 . Since

$$I_0 = \Psi(r = \infty), \quad \frac{\Lambda}{3} I_1 = \frac{d\Psi}{dr_\star}(r = \infty), \quad (3.16)$$

it follows that general boundary conditions (also called mixed or Robin) can be expressed in terms of the ratio

$$\frac{I_0}{I_1} = \gamma \quad (3.17)$$

for fixed constant γ that can assume all values, including zero and infinity. Thus, the allowed spectrum of frequencies ω obeys a transcendental relation given by ratios of gamma functions and can only be solved numerical for general values of γ . In all cases, however, the frequencies come in pairs $(\omega, -\omega)$, as can be readily seen from the particular expressions of I_0 and I_1 in terms of products of gamma functions. Consequently, by appropriate superposition of them, the metric perturbations can always be taken real.

There are two special boundary conditions that yield simple spectrum of perturbations. First, we consider Dirichlet boundary conditions, which are favored in most applications in AdS/CFT correspondence, by imposing $I_0 = 0$. Since the gamma function blows up only when its argument assumes the values $0, -1, -2, \dots$, it follows that the quantization condition for the frequencies is

$$\sqrt{-\frac{3}{\Lambda}} \omega_D = \pm(2n + l + 2) \quad (3.18)$$

with $n = 0, 1, 2, \dots$. Second, we may also consider Neumann boundary conditions by imposing $I_1 = 0$. In this case, the corresponding quantization condition for the frequencies takes the form

$$\sqrt{-\frac{3}{\Lambda}} \omega_N = \pm(2n + l + 1) \quad (3.19)$$

with $n = 0, 1, 2, \dots$. Note that for given l , the frequencies that yield $I_0 = 0$ have $I_1 \neq 0$ and those that yield $I_1 = 0$ have $I_0 \neq 0$. We also note that ω_N follows from ω_D by letting $l \rightarrow l - 1$.

The effective Schrödinger potential derives from a superpotential $W(x)$,

$$W(x) = l \cot x , \quad (3.20)$$

since

$$V_{\mp}(x) = W^2(x) \mp \frac{d}{dx}W(x) = \frac{l(l \pm 1)}{\sin^2 x} - l^2 . \quad (3.21)$$

These are supersymmetric partner potentials which happen to follow from each other by the simple substitution $l \rightarrow l - 1$. The energy levels E of $V_{\mp}(x)$ provide the allowed frequencies of perturbation,

$$E = -\frac{3}{\Lambda} \omega^2 - l^2 , \quad (3.22)$$

and the corresponding wave-functions $\Psi_{\mp}(x)$ with the same E (and hence the same ω) are interrelated as

$$\left(\mp \frac{d}{dx} + W(x) \right) \Psi_{\pm}(x) = \sqrt{E} \Psi_{\mp}(x) \quad (3.23)$$

by supersymmetric quantum mechanics, [14]. In turn, this implies, setting $x = \pi/2$, that

$$\mp \frac{d}{dx} \Psi_{\pm}(\pi/2) = \sqrt{E} \Psi_{\mp}(\pi/2) , \quad (3.24)$$

exchanging Dirichlet and Neumann boundary conditions under $l \rightarrow l - 1$ for the particular Schrödinger problem. This explains the relation among ω_D and ω_N noted earlier.

More generally, mixed boundary conditions with parameter γ , as defined above, are mapped to mixed boundary conditions with parameter

$$\gamma \rightarrow \frac{\Lambda}{3E} \cdot \frac{1}{\gamma} \quad (3.25)$$

under $l \rightarrow l - 1$, so that a given frequency ω appears in the spectra of both $V_{\mp}(x)$. This relation is also apparent from the particular form of the coefficients I_0 and I_1 above, using the identity $\Gamma(z + 1) = z\Gamma(z)$, but has no special meaning in the holographic description of the perturbed space-time, as far as we can tell now.

For fixed l , the gravitational perturbations of AdS_4 space-time can satisfy arbitrary boundary conditions at spatial infinity. This possibility was investigated in the literature before, following the thorough analysis of ref. [4], and it was found that all boundary conditions are allowed as they are in one-to-one correspondence with the self-adjoint extensions of the positive definite operator

$$L = -\frac{d^2}{dx^2} + \frac{l(l + 1)}{\sin^2 x} . \quad (3.26)$$

Summarizing, the boundary conditions for axial and polar perturbations of AdS_4 space-time are independent from each other, and so is the spectrum of allowed frequencies, whereas the effective Schrödinger equations are the same. Thus, for fixed l , the axial and polar perturbations are isospectral provided that they both satisfy the same boundary conditions. It should be contrasted to the situation of AdS_4 black holes whose axial and polar perturbations obey supersymmetric partner potential Schrödinger equations for any given l , [11]. In the latter case, axial and polar perturbations will be isospectral if and only if the corresponding boundary conditions are supersymmetric partners.

4 Energy-momentum tensor

The energy-momentum tensor on asymptotically AdS spaces M is defined by varying the gravitational action S_{gr} with respect to the boundary metric γ on ∂M , as

$$T^{ab} = \frac{2}{\sqrt{-\det\gamma}} \frac{\delta S_{\text{gr}}}{\delta \gamma_{ab}} . \quad (4.1)$$

The resulting expression typically diverge, but it is always possible by holographic renormalization to obtain finite results adding an appropriately chosen boundary counter-term whose form depends on the dimensionality of space-time, [15], [16], [17].

In AdS_4 spaces, in particular, the gravitational action consists of bulk and boundary terms chosen as follows,

$$\begin{aligned} S_{\text{gr}} = & -\frac{1}{2\kappa^2} \int_M d^4x \sqrt{-\det g} (R[g] + 2\Lambda) - \frac{1}{\kappa^2} \int_{\partial M} d^3x \sqrt{-\det\gamma} K \\ & - \frac{2}{\kappa^2} \sqrt{-\frac{\Lambda}{3}} \int_{\partial M} d^3x \sqrt{-\det\gamma} \left(1 + \frac{3}{4\Lambda} R[\gamma] \right) . \end{aligned} \quad (4.2)$$

The first boundary contribution is the usual Gibbons-Hawking term written in terms of the trace of the second fundamental form, i.e., the extrinsic mean curvature, $K = \gamma^{ab} K_{ab}$, associated to the embedding of ∂M in M . The second boundary contribution is the contact term needed to remove all divergencies in the present case. Then, according to definition, the energy-momentum tensor is expressed in terms of the intrinsic and extrinsic geometry of the AdS boundary at infinity, prior to rescaling, as

$$\kappa^2 T_{ab} = K_{ab} - K \gamma_{ab} - 2 \sqrt{-\frac{\Lambda}{3}} \gamma_{ab} + \sqrt{-\frac{3}{\Lambda}} \left(R_{ab}[\gamma] - \frac{1}{2} R[\gamma] \gamma_{ab} \right) . \quad (4.3)$$

The computation is performed by first writing the metric g on M in the form

$$ds^2 = N^2 dr^2 + \gamma_{ab} (dx^a + N^a dr) (dx^b + N^b dr) \quad (4.4)$$

using appropriately chosen (N, N^a) functions. The three-dimensional surface arising at fixed distance r serves as boundary ∂M_r to the interior four-dimensional region M_r . The induced metric on ∂M_r is γ_{ab} evaluated at the boundary value of r , which is held finite at this point. The second fundamental form K_{ab} on ∂M_r is defined using the outward pointing normal vector η_μ to the boundary ∂M_r with components $\eta_\mu = N \delta_\mu^r$. In particular, one has

$$K_{ab} = -\nabla_{(a} \eta_{b)} = N \Gamma_{ab}^r[g] . \quad (4.5)$$

At the end of the computation, T^{ab} on the AdS boundary ∂M is obtained by letting $r \rightarrow \infty$.

The boundary metric acquires an infinite Weyl factor as r is taken to infinity, and, therefore, it is more appropriate to think of the AdS boundary as a conformal class of boundaries and define \mathcal{S} as the boundary space-time with metric

$$ds_{\mathcal{S}}^2 = \lim_{r \rightarrow \infty} \left(-\frac{3}{\Lambda r^2} \gamma_{ab} dx^a dx^b \right) . \quad (4.6)$$

Then, the renormalized energy-momentum tensor on \mathcal{J} is defined accordingly by

$$T_{ab}^{\text{renorm}} = \lim_{r \rightarrow \infty} \left(\sqrt{-\frac{\Lambda}{3}} r T_{ab} \right) \quad (4.7)$$

and it is finite, traceless and conserved. This is the quantity that will be computed for all different type of gravitational perturbations of AdS_4 space-time. The decorations will be dropped in the sequel to simplify the notation.

(i). **Axial perturbations** : Applying the holographic renormalization method to axial perturbations of AdS_4 space-time is straightforward. The boundary data depends on the coefficients I_0 and I_1 in the asymptotic expansion of the effective wave-function in powers of $1/r$,

$$\Psi_{\text{RW}}(r) = I_0 + \frac{I_1}{r} + \frac{I_2}{r^2} + \frac{I_3}{r^3} + \dots, \quad (4.8)$$

since all subleading terms are fixed uniquely by the choice of boundary conditions. Since the calculation is quite involved, we only present the final result for the energy-momentum tensor and metric at the conformal boundary when arbitrary boundary conditions are imposed at spatial infinity.

The three-dimensional metric on \mathcal{J} takes the following form, after conformal rescaling,

$$ds_{\mathcal{J}}^2 = -dt^2 - \frac{3}{\Lambda} (d\theta^2 + \sin^2\theta d\phi^2) + 2 \frac{iI_0}{\omega} e^{-i\omega t} \sin\theta \partial_{\theta} P_l(\cos\theta) dt d\phi \quad (4.9)$$

and the non-vanishing components of the renormalized energy-momentum tensor for axial perturbations of AdS_4 space-time turn out to be

$$\kappa^2 T_{t\phi} = \frac{i\Lambda}{6\omega} (l-1)(l+2) I_1 e^{-i\omega t} \sin\theta \partial_{\theta} P_l(\cos\theta), \quad (4.10)$$

$$\kappa^2 T_{\theta\phi} = -\frac{1}{2} I_1 e^{-i\omega t} \sin\theta [l(l+1) P_l(\cos\theta) + 2\cot\theta \partial_{\theta} P_l(\cos\theta)]. \quad (4.11)$$

It can be easily verified that this energy-momentum tensor is traceless and conserved on \mathcal{J} . It vanishes only when $I_1 = 0$, i.e., when Ψ_{RW} satisfies Neumann boundary conditions at spatial infinity. At the same time, the boundary metric depends on time and it becomes static only when $I_0 = 0$, i.e., when Ψ_{RW} satisfies Dirichlet boundary conditions at spatial infinity.

(ii). **Polar perturbations** : The computation of the energy-momentum tensor for polar perturbations of AdS_4 space-time is much more involved. It requires one more term in the asymptotic expansion of Ψ_Z ,

$$\Psi_Z(r) = J_0 + \frac{J_1}{r} + \frac{J_2}{r^2} + \frac{J_3}{r^3} + \frac{J_4}{r^4} + \dots, \quad (4.12)$$

but all boundary data is fully determined by J_0 and J_1 as before. Here, we use the symbols J_i to distinguish from the symbols I_i appearing as coefficients in the asymptotic expansion of the axial wave-functions. We also skip the intermediate details and present

the final result for the energy-momentum tensor and metric at the conformal boundary when arbitrary boundary conditions are imposed at spatial infinity.

In this case, the three-dimensional metric on the boundary takes the following form, after conformal rescaling,

$$ds_{\mathcal{J}}^2 = -dt^2 - \frac{3}{\Lambda} \left[1 + \frac{\Lambda}{3} J_1 e^{-i\omega t} P_l(\cos\theta) \right] (d\theta^2 + \sin^2\theta d\phi^2) . \quad (4.13)$$

Explicit calculation shows that the non-vanishing components of the renormalized energy-momentum tensor for polar perturbations of AdS_4 space-time are:

$$\kappa^2 T_{tt} = -\frac{\Lambda}{12} (l-1)l(l+1)(l+2) J_0 e^{-i\omega t} P_l(\cos\theta) , \quad (4.14)$$

$$\begin{aligned} \kappa^2 T_{\theta\theta} = & -\frac{1}{4} l(l+1) \left(1 + \frac{3\omega^2}{\Lambda} \right) J_0 e^{-i\omega t} P_l(\cos\theta) \\ & -\frac{1}{4} \left(l(l+1) + \frac{6\omega^2}{\Lambda} \right) J_0 e^{-i\omega t} \cot\theta \partial_\theta P_l(\cos\theta), \end{aligned} \quad (4.15)$$

$$\begin{aligned} \kappa^2 T_{\phi\phi} = & \frac{1}{4} l(l+1) \left(l(l+1) - 1 + \frac{3\omega^2}{\Lambda} \right) J_0 e^{-i\omega t} \sin^2\theta P_l(\cos\theta) + \\ & \frac{1}{4} \left(l(l+1) + \frac{6\omega^2}{\Lambda} \right) J_0 e^{-i\omega t} \sin\theta \cos\theta \partial_\theta P_l(\cos\theta) , \end{aligned} \quad (4.16)$$

$$\kappa^2 T_{t\theta} = \frac{1}{4} i\omega (l-1)(l+2) J_0 e^{-i\omega t} \partial_\theta P_l(\cos\theta) . \quad (4.17)$$

It can be verified, as consistency check, that the energy-momentum tensor is traceless and conserved on \mathcal{J} . It vanishes only when $J_0 = 0$, i.e., when Ψ_Z satisfies Dirichlet boundary conditions at spatial infinity. At the same time, the boundary metric depends on time and it becomes static only when $J_1 = 0$, i.e., when Ψ_Z satisfies Neumann boundary conditions at spatial infinity. This behavior is complementary to the axial perturbations studied above and calls for an explanation.

5 Cotton tensor duality

When the boundary metric is static it is also conformally flat, since it corresponds to $2+1$ dimensional Einstein universe

$$ds_{\mathcal{J}}^2 = -dt^2 - \frac{3}{\Lambda} (d\theta^2 + \sin^2\theta d\phi^2) . \quad (5.1)$$

Otherwise, for general boundary conditions, the boundary metric varies with time. A way to measure the deviation from the conformally flat form is provided by the Weyl tensor of the metric, but in three space-time dimensions all components vanish identically. In this case, however, there is an alternative way provided by the so called Cotton tensor of the metric.

The Cotton tensor of a three-dimensional metric γ_{ab} is defined as follows,

$$\begin{aligned} C^{ab} &= \frac{1}{2\sqrt{-\det\gamma}} \left(\epsilon^{acd} \nabla_c R^b_d + \epsilon^{bcd} \nabla_c R^a_d \right) \\ &= \frac{\epsilon^{acd}}{\sqrt{-\det\gamma}} \nabla_c \left(R^b_d - \frac{1}{4} \delta^b_d R \right) \end{aligned} \quad (5.2)$$

and has odd parity. The density $\sqrt{\det\gamma} C^a_b$ remains invariant under local conformal changes and vanishes if and only if the metric is conformally flat. The Cotton tensor is symmetric, traceless and identically covariantly conserved. It provides the energy-momentum tensor of the three-dimensional gravitational Chern-Simons theory, [18],

$$C_{ab} = \frac{1}{\sqrt{-\det\gamma}} \frac{\delta S_{\text{CS}}}{\delta \gamma^{ab}} \quad (5.3)$$

with action

$$S_{\text{CS}} = \frac{1}{2} \int d^3x \sqrt{-\det\gamma} \epsilon^{abc} \Gamma_{ae}^d \left(\partial_b \Gamma_{cd}^e + \frac{2}{3} \Gamma_{bf}^e \Gamma_{cd}^f \right). \quad (5.4)$$

Let us now compute the Cotton tensor for the boundary metric of perturbed AdS_4 space-time, setting $\epsilon^{t\theta\phi} = 1$. We study separately the axial and polar perturbations satisfying arbitrary boundary conditions.

(i). Axial perturbations : In this case, the boundary metric takes the form

$$ds_{\mathcal{J}}^2(\text{axial}) = -dt^2 - \frac{3}{\Lambda} \left(d\theta^2 + \sin^2\theta d\phi^2 \right) + 2 \frac{iI_0}{\omega} e^{-i\omega t} \sin\theta \partial_\theta P_l(\cos\theta) dt d\phi \quad (5.5)$$

when general boundary conditions with coefficients I_0 and I_1 are imposed at spatial infinity on the axial wave-function.

Then, explicit computation shows that the non-vanishing components of the corresponding Cotton tensor are

$$C_{tt} = \frac{i\Lambda^2}{18\omega} (l-1)l(l+1)(l+2) I_0 e^{-i\omega t} P_l(\cos\theta), \quad (5.6)$$

$$\begin{aligned} C_{\theta\theta} &= \frac{i\Lambda}{6\omega} l(l+1) \left(1 + \frac{3\omega^2}{\Lambda} \right) I_0 e^{-i\omega t} P_l(\cos\theta) + \\ &\quad \frac{i\Lambda}{6\omega} \left(l(l+1) + \frac{6\omega^2}{\Lambda} \right) I_0 e^{-i\omega t} \cot\theta \partial_\theta P_l(\cos\theta), \end{aligned} \quad (5.7)$$

$$\begin{aligned} C_{\phi\phi} &= -\frac{i\Lambda}{6\omega} l(l+1) \left(l(l+1) - 1 + \frac{3\omega^2}{\Lambda} \right) I_0 e^{-i\omega t} \sin^2\theta P_l(\cos\theta) \\ &\quad - \frac{i\Lambda}{6\omega} \left(l(l+1) + \frac{6\omega^2}{\Lambda} \right) I_0 e^{-i\omega t} \sin\theta \cos\theta \partial_\theta P_l(\cos\theta), \end{aligned} \quad (5.8)$$

$$C_{t\theta} = \frac{\Lambda}{6} (l-1)(l+2) I_0 e^{-i\omega t} \partial_\theta P_l(\cos\theta). \quad (5.9)$$

and coincide with the components of the energy-momentum tensor for polar perturbations of AdS_4 space-time satisfying general boundary conditions with respective coefficients J_0 and J_1 , provided that

$$J_0 = -i\frac{2\Lambda}{3\omega}I_0 . \quad (5.10)$$

The match is exact provided that the same frequencies ω arise for the axial and polar perturbations, i.e., when the same boundary conditions are imposed at spatial infinity,

$$\frac{I_0}{I_1} = \frac{J_0}{J_1} . \quad (5.11)$$

Then, we have the following relation among the two distinct type of perturbations satisfying the same general boundary conditions,

$$C_{ab}(\text{axial}) = \kappa^2 T_{ab}(\text{polar}) . \quad (5.12)$$

(ii). **Polar perturbations** : In this case, the boundary metric takes the form

$$ds_{\mathcal{S}}^2(\text{polar}) = -dt^2 - \frac{3}{\Lambda} \left[1 + \frac{\Lambda}{3} J_1 e^{-i\omega t} P_l(\cos\theta) \right] (d\theta^2 + \sin^2\theta d\phi^2) . \quad (5.13)$$

when general boundary conditions with coefficients J_0 and J_1 are imposed at spatial infinity on the polar wave-function.

Again, explicit computation shows that the non-vanishing components of the corresponding Cotton tensor are

$$C_{t\phi} = \frac{\Lambda^2}{36} (l-1)(l+2) J_1 e^{-i\omega t} \sin\theta \partial_\theta P_l(\cos\theta) , \quad (5.14)$$

$$C_{\theta\phi} = \frac{\Lambda}{12} i\omega J_1 e^{-i\omega t} \sin\theta [l(l+1) P_l(\cos\theta) + 2\cot\theta \partial_\theta P_l(\cos\theta)] . \quad (5.15)$$

and coincide with the components of the energy-momentum tensor for axial perturbations of AdS_4 space-time satisfying general boundary conditions with respective coefficients I_0 and I_1 , provided that

$$I_1 = -\frac{\Lambda}{6} i\omega J_1 . \quad (5.16)$$

As before, the match is exact provided that the same frequencies ω arise for the axial and polar perturbations, i.e., when the same boundary conditions (5.11) are imposed on both sectors at spatial infinity. Then, under this condition, we obtain the following general relation connecting the two perturbations,

$$C_{ab}(\text{polar}) = \kappa^2 T_{ab}(\text{axial}) . \quad (5.17)$$

Note at this end that the situation changes drastically in the presence of black holes in AdS_4 space-time, [7]. First of all, the holographic energy-momentum tensor of the AdS_4 Schwarzschild solution does not vanish, as its components depend on the mass parameter. Also, according to the general theory of quasi-normal modes in four space-time

dimensions, axial and polar perturbations of the Schwarzschild solution satisfy different Schrödinger equations, which are usually referred as Regge-Wheeler and Zerilli equations, respectively. However, the two effective potentials are partners as in supersymmetric quantum mechanics, [14], and, as a result, there is still a relation between the energy-momentum tensor of perturbed black holes and the Cotton tensor of a boundary dual metric as given by

$$C_{ab}(\text{axial}) = \kappa^2 \delta T_{ab}(\text{polar}) , \quad C_{ab}(\text{polar}) = \kappa^2 \delta T_{ab}(\text{axial}) . \quad (5.18)$$

It was noted before, [7], that this relation is only valid when the axial and polar perturbations satisfy specific supersymmetric partner boundary conditions that encompass the hydrodynamic modes of black holes. The result should be contrasted with the energy-momentum/Cotton tensor duality for perturbed AdS_4 space-time, which, as described above, is valid for all boundary conditions provided that they are the same in both sectors.

6 Electric/magnetic duality in linearized gravity

Free gauge fields in four space-time dimensions exhibit two physical degrees of freedom that can be rotated into one another by a canonical transformation mixing the two pairs of unconstrained dynamical variables, while keeping the Hamiltonian form-invariant. It is a general result that extends the electric/magnetic duality of electromagnetism to other physical fields including Einstein gravity. For gravity, the duality is defined at the linear level by considering small perturbations around a reference metric, $g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}$. There is a non-local transformation to perturbations around the same reference metric $\tilde{g}_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta \tilde{g}_{\mu\nu}$, acting on the space of solutions of vacuum Einstein equation and which can also be realized as symmetry of the gravitational action. It generalizes the Ehlers transformation¹ to metrics that do not necessarily admit Killing isometries, but it breaks down at first self-interacting cubic approximation to general relativity, [22], which, however, is not relevant to the present work. The reference metric $g_{\mu\nu}^{(0)}$ is not arbitrary in this study; it is provided by the metric of flat Minkowski space-time when the cosmological constant Λ vanishes and by the metric of $(A)dS_4$ space-time when $\Lambda \neq 0$.

The electric/magnetic duality of linearized gravity was initially formulated for $\Lambda = 0$, [23], but it extends rather easily to vacuum Einstein equations with cosmological constant,

¹The space of vacuum solutions of Einstein field equations with a Killing isometry in four space-time dimensions admit the action of an $SL(2, R)$ group known as Ehlers symmetry, [19]. This group acts as solution generating symmetry and it is present at the full non-linear level only when $\Lambda = 0$. It arises by rewriting the four-dimensional Einstein-Hilbert action as three-dimensional gravity coupled to an $SL(2, R)/U(1)$ non-linear sigma model. An $SO(2)$ subgroup acts continuously as S -duality and interchanges “bolts” and “nuts” in space-time as in electric/magnetic duality, [20] (but see also ref. [21]). The Ehlers symmetry breaks down in the presence of cosmological constant, unlike the electric/magnetic duality of linearized gravity that holds for all values of Λ .

[24], [25], [26]; see also ref. [27] for earlier important work on the same subject. Here, we choose to work with $\Lambda < 0$, although the description below is valid for all values of the cosmological constant. The key quantity is provided by

$$Z_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{\Lambda}{3}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \quad (6.1)$$

that arises by restricting the Weyl curvature tensor in four space-time dimensions to on-shell metrics, [25]. Clearly, it fulfills the identities

$$Z_{\mu[\nu\rho\sigma]} = 0, \quad \nabla_{[\lambda}Z_{\mu\nu]\rho\sigma} = 0, \quad (6.2)$$

and the on-shell metrics satisfy the equation

$$Z^\rho{}_{\mu\rho\nu} \equiv Z_{\mu\nu} = 0, \quad (6.3)$$

which is equivalent to $R_{\mu\nu} = \Lambda g_{\mu\nu}$. One also defines the dual curvature tensor

$$\tilde{Z}_{\mu\nu\rho\sigma} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\kappa\lambda}Z_{\kappa\lambda\rho\sigma}, \quad (6.4)$$

which fulfills similar identities, but with reverse meaning,

$$\tilde{Z}_{\mu[\nu\rho\sigma]} = 0, \quad \nabla_{[\lambda}\tilde{Z}_{\mu\nu]\rho\sigma} = 0 \quad (6.5)$$

and it satisfies the classical equation of motion

$$\tilde{Z}^\rho{}_{\mu\rho\nu} \equiv \tilde{Z}_{\mu\nu} = 0. \quad (6.6)$$

Here, $\epsilon_{\mu\nu}{}^{\kappa\lambda}$ is the covariant fully antisymmetric symbol in four-dimensional space-time with $\epsilon_{rt\theta\phi} = \sqrt{-\det g}$.

Linearized gravity around AdS_4 space-time exhibits a duality that exchanges Bianchi identities with the classical equations of motion, as in electromagnetism, by

$$Z'_{\mu\nu\rho\sigma} = \tilde{Z}_{\mu\nu\rho\sigma}, \quad \tilde{Z}'_{\mu\nu\rho\sigma} = -Z_{\mu\nu\rho\sigma}. \quad (6.7)$$

Actually, at the linear level, it is appropriate to replace the covariant derivatives ∇ by ordinary derivatives ∂ . Also, it is useful to introduce the electric and magnetic components of the Weyl tensor as

$$\mathcal{E}_{ab} = Z_{arbr}, \quad \mathcal{B}_{ab} = \tilde{Z}_{arbr}, \quad (6.8)$$

using the radial coordinate r of AdS_4 space-time. Then, the gravitational duality transformation is realized as

$$\mathcal{E}'_{ab} = \mathcal{B}_{ab}, \quad \mathcal{B}'_{ab} = -\mathcal{E}_{ab}, \quad (6.9)$$

and, more generally, one may consider an $SO(2)$ rotation of these components parametrized by an arbitrary angle δ ,

$$\begin{pmatrix} \mathcal{E}'_{ab} \\ \mathcal{B}'_{ab} \end{pmatrix} = \begin{pmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{pmatrix} \begin{pmatrix} \mathcal{E}_{ab} \\ \mathcal{B}_{ab} \end{pmatrix}. \quad (6.10)$$

Details of the proof can be found in the original works showing that the duality is also a symmetry of the linearized action (see, in particular, ref. [23] and [25]).

The electric and magnetic tensors are represented by 3×3 symmetric traceless matrices on-shell, and, as such, they have five independent components each. Let us compute them for the axial and polar perturbations of AdS_4 space-time and show that their interchange accounts for the duality (6.9).

(i). **Axial perturbations** : Taking into account the general form of the axial perturbations around AdS_4 space-time and their explicit dependence on the frequencies ω and the wave-functions Ψ_{RW} , we find the following results on-shell for the electric components

$$\mathcal{E}_{\theta\phi}^{\text{axial}} = -\frac{\cos^3 x}{2} \frac{d}{dx} (\sin x \Psi_{RW}(x)) e^{-i\omega t} \sin\theta [(l(l+1) + 2\cot\theta \partial_\theta) P_l(\cos\theta)] , \quad (6.11)$$

$$\mathcal{E}_{t\phi}^{\text{axial}} = \frac{i\Lambda}{6\omega} (l-1)(l+2) \frac{\cos^3 x}{\sin x} \left(\frac{d}{dx} \Psi_{RW}(x) \right) e^{-i\omega t} \sin\theta \partial_\theta P_l(\cos\theta) , \quad (6.12)$$

and similarly for the magnetic components

$$\mathcal{B}_{tt}^{\text{axial}} = \frac{i}{2\omega} \left(\sqrt{-\frac{\Lambda}{3}} \right)^3 (l-1)l(l+1)(l+2) \frac{\cos^3 x}{\sin^3 x} \Psi_{RW}(x) e^{-i\omega t} P_l(\cos\theta) , \quad (6.13)$$

$$\begin{aligned} \mathcal{B}_{\theta\theta}^{\text{axial}} = & -\frac{i}{2\omega} \sqrt{-\frac{\Lambda}{3}} l(l+1) \frac{\cos^3 x}{\sin x} \left[\left(1 + \frac{3\omega^2}{\Lambda} \sin^2 x \right) \Psi_{RW}(x) + \right. \\ & \left. \sin x \cos x \frac{d}{dx} \Psi_{RW}(x) \right] e^{-i\omega t} P_l(\cos\theta) \\ & -\frac{i}{\omega} \sqrt{-\frac{\Lambda}{3}} \frac{\cos^3 x}{\sin x} \left[\left(\frac{l(l+1)}{2} + \frac{3\omega^2}{\Lambda} \sin^2 x \right) \Psi_{RW}(x) + \right. \\ & \left. \sin x \cos x \frac{d}{dx} \Psi_{RW}(x) \right] e^{-i\omega t} \cot\theta \partial_\theta P_l(\cos\theta) , \end{aligned} \quad (6.14)$$

$$\begin{aligned} \mathcal{B}_{\phi\phi}^{\text{axial}} = & \frac{i}{2\omega} \sqrt{-\frac{\Lambda}{3}} l(l+1) \frac{\cos^3 x}{\sin x} \left[\left(l^2 + l - 1 + \frac{3\omega^2}{\Lambda} \sin^2 x \right) \Psi_{RW}(x) + \right. \\ & \left. \sin x \cos x \frac{d}{dx} \Psi_{RW}(x) \right] e^{-i\omega t} \sin^2\theta P_l(\cos\theta) \\ & +\frac{i}{\omega} \sqrt{-\frac{\Lambda}{3}} \frac{\cos^3 x}{\sin x} \left[\left(\frac{l(l+1)}{2} + \frac{3\omega^2}{\Lambda} \sin^2 x \right) \Psi_{RW}(x) + \right. \\ & \left. \sin x \cos x \frac{d}{dx} \Psi_{RW}(x) \right] e^{-i\omega t} \sin\theta \cos\theta \partial_\theta P_l(\cos\theta) , \end{aligned} \quad (6.15)$$

$$\mathcal{B}_{t\theta}^{\text{axial}} = -\frac{1}{2} \sqrt{-\frac{\Lambda}{3}} (l-1)(l+2) \frac{\cos^3 x}{\sin x} \Psi_{RW}(x) e^{-i\omega t} \partial_\theta P_l(\cos\theta) . \quad (6.16)$$

Here, we have also used that the metric of axially perturbed space-time has determinant

$$\sqrt{-\det g} = r^2 \sin\theta = -\frac{3}{\Lambda} \tan^2 x \sin\theta . \quad (6.17)$$

(ii). **Polar perturbations** : Performing the same calculation for the polar perturbations of AdS_4 space-time, using their explicit dependence on the corresponding frequencies ω and the wave-functions Ψ_Z , we find the following results on-shell for the electric components

$$\mathcal{E}_{tt}^{\text{polar}} = -\frac{1}{4} \left(\sqrt{-\frac{\Lambda}{3}} \right)^3 (l-1)l(l+1)(l+2) \frac{\cos^3 x}{\sin^3 x} \Psi_Z(x) e^{-i\omega t} P_l(\cos\theta) , \quad (6.18)$$

$$\begin{aligned} \mathcal{E}_{\theta\theta}^{\text{polar}} = & \frac{1}{4} \sqrt{-\frac{\Lambda}{3}} l(l+1) \frac{\cos^3 x}{\sin x} \left[\left(1 + \frac{3\omega^2}{\Lambda} \sin^2 x \right) \Psi_Z(x) + \right. \\ & \left. \sin x \cos x \frac{d}{dx} \Psi_Z(x) \right] e^{-i\omega t} P_l(\cos\theta) \\ & + \frac{1}{2} \sqrt{-\frac{\Lambda}{3}} \frac{\cos^3 x}{\sin x} \left[\left(\frac{l(l+1)}{2} + \frac{3\omega^2}{\Lambda} \sin^2 x \right) \Psi_Z(x) + \right. \\ & \left. \sin x \cos x \frac{d}{dx} \Psi_Z(x) \right] e^{-i\omega t} \cot\theta \partial_\theta P_l(\cos\theta) , \end{aligned} \quad (6.19)$$

$$\begin{aligned} \mathcal{E}_{\phi\phi}^{\text{polar}} = & -\frac{1}{4} \sqrt{-\frac{\Lambda}{3}} l(l+1) \frac{\cos^3 x}{\sin x} \left[\left(l^2 + l - 1 + \frac{3\omega^2}{\Lambda} \sin^2 x \right) \Psi_Z(x) + \right. \\ & \left. \sin x \cos x \frac{d}{dx} \Psi_Z(x) \right] e^{-i\omega t} \sin^2\theta P_l(\cos\theta) \\ & - \frac{1}{2} \sqrt{-\frac{\Lambda}{3}} \frac{\cos^3 x}{\sin x} \left[\left(\frac{l(l+1)}{2} + \frac{3\omega^2}{\Lambda} \sin^2 x \right) \Psi_Z(x) + \right. \\ & \left. \sin x \cos x \frac{d}{dx} \Psi_Z(x) \right] e^{-i\omega t} \sin\theta \cos\theta \partial_\theta P_l(\cos\theta) , \end{aligned} \quad (6.20)$$

$$\mathcal{E}_{t\theta}^{\text{polar}} = -\frac{i\omega}{4} \sqrt{-\frac{\Lambda}{3}} (l-1)(l+2) \frac{\cos^3 x}{\sin x} \Psi_Z(x) e^{-i\omega t} \partial_\theta P_l(\cos\theta) , \quad (6.21)$$

and similarly for the magnetic components

$$\mathcal{B}_{\theta\phi}^{\text{polar}} = i\omega \frac{\cos^3 x}{4} \frac{d}{dx} (\sin x \Psi_Z(x)) e^{-i\omega t} \sin\theta [(l(l+1) + 2\cot\theta \partial_\theta) P_l(\cos\theta)] , \quad (6.22)$$

$$\mathcal{B}_{t\phi}^{\text{polar}} = \frac{\Lambda}{12} (l-1)(l+2) \frac{\cos^3 x}{\sin x} \left(\frac{d}{dx} \Psi_Z(x) \right) e^{-i\omega t} \sin\theta \partial_\theta P_l(\cos\theta) . \quad (6.23)$$

Here, the determinant of the metric of perturbed space-time is given by

$$\sqrt{-\det g} = r^2 \sin\theta \left(1 + K(r) e^{-i\omega t} P_l(\cos\theta) \right) , \quad (6.24)$$

which, in turn, can be written in terms of the angular variable x , instead of r , used in the computations.

Since both perturbations satisfy the same Schrödinger equation, we have $\Psi_{\text{RW}} = \Psi_Z$ (up to an arbitrary multiplicative constant) when the same boundary conditions are

imposed at infinity; they also ensure that the frequencies ω are the same in both cases. In fact, choosing the multiplicative constant as

$$\Psi_{\text{RW}}(x) = \frac{i\omega}{2} \Psi_Z(x) \quad (6.25)$$

the result of the calculation is neatly summarized as

$$\mathcal{E}_{ab}^{\text{polar}} = \mathcal{B}_{ab}^{\text{axial}}, \quad \mathcal{B}_{ab}^{\text{polar}} = -\mathcal{E}_{ab}^{\text{axial}} \quad (6.26)$$

showing that the gravitational duality (6.9) is realized by exchanging axial with polar perturbations, as advertised above.

This simple fact has not been spelled in the literature so far, to the best of our knowledge, since gravitational duality was only considered as abstract transformation acting non-locally on the perturbations $\delta g_{\mu\nu}$. It is also obviously valid for all values of the cosmological constant. Recall, for this purpose, that $x = r_\star \sqrt{-\Lambda/3}$ and, therefore, the effective Schrödinger problem takes the following form in the limit $\Lambda = 0$,

$$\left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} \right) \Psi(r) = \omega^2 \Psi(r) \quad (6.27)$$

in terms of the radial variable r ($= r_\star$), instead of x . Then, \mathcal{E}_{ab} and \mathcal{B}_{ab} assume their respective values obtained from the Λ -dependent expressions above, which are well defined and involve $\Psi(r)$ and its derivatives with respect to r . Thus, for $\Lambda = 0$, the duality of linearized gravity is also realized by interchanging axial and polar perturbations of flat space-time, without ever using the form of the wave-functions $\Psi(r)$. Likewise, for perturbations of dS_4 space-time, one simply has to consider the analytic continuation of the trigonometric functions of x into their hyperbolic counterparts and reach the same conclusion. The exchange of axial and polar perturbations resembles the exchange of “nuts” and “bolts” under the action of Ehlers symmetry, where it is appropriate to use.

The holographic manifestation in AdS_4 space-time is the energy-momentum/Cotton tensor duality at the conformal boundary that gives rise to the dual graviton correspondence. This is made more precise by further noticing

$$\lim_{r \rightarrow \infty} \left(\frac{\Lambda}{3} r^3 \mathcal{E}_{ab} \right) = \kappa^2 T_{ab}, \quad \lim_{r \rightarrow \infty} \left(\frac{\Lambda^2}{9} r^3 \mathcal{B}_{ab} \right) = C_{ab} \quad (6.28)$$

for either type of perturbation, using the asymptotic expansion of the effective Schrödinger wave-functions, and completes the proof of the main assertion for perturbations of AdS_4 .

7 Conclusions

We have obtained a precise realization of electric/magnetic duality in linearized gravity in terms of the interchange of axial and polar perturbations around AdS_4 space-time. Since a generic perturbation can be decomposed into a sum of axial and polar parts, our

results “trivialize” in a certain sense the action of dualities. We have also shown that the holographic manifestation of this duality is the energy-momentum/Cotton tensor duality that realizes the dual graviton correspondence in AdS_4/CFT_3 under general boundary conditions. These results should be taken into account in future work to investigate the structure of the three-dimensional field theory at the conformal boundary.

When a black hole is added in space-time the situation changes. It is not known whether the duality of linearized gravity persists for perturbations of the Schwarzschild solution. Still one has two distinct type of perturbations, axial and polar, as in AdS_4 space-time, but they do not satisfy the same effective Schrödinger equation. They rather satisfy supersymmetric partner Schrödinger equations, irrespective of Λ , as noted before. It is natural to expect that this partnership among axial and polar perturbations can also be understood in terms of gravitational duality when appropriately formulated on black hole backgrounds. This possibility is currently under investigation and hopefully will explain why supersymmetric quantum mechanics is at work in the four-dimensional theory of quasi-normal modes of black holes for all values of Λ . Perhaps that is the best one can do for perturbations of non-trivial gravitational backgrounds. In turn, it may also provide a deeper understanding of the energy-momentum/Cotton tensor duality for perturbations of AdS_4 black holes with appropriate boundary conditions on the axial and polar sectors of the correspondence, [7].

We also note that there are special configurations (other than the cases we are considering here) which satisfy the self-duality relations

$$\tilde{Z}_{\mu\nu\rho\sigma}(g) = \pm Z_{\mu\nu\rho\sigma}(\tilde{g}) \quad (7.1)$$

with $g_{\mu\nu} = \tilde{g}_{\mu\nu}$. They define the so called Λ -instantons, in the nomenclature of ref. [25], whose energy-momentum tensor equals the Cotton tensor of their boundary metric γ_{ab} , when $\Lambda < 0$, [28] (but see also ref. [29]),

$$\kappa^2 T_{ab}(\text{instantons}) = C_{ab}(\gamma) . \quad (7.2)$$

It will be interesting to examine the fate of gravitational duality rotations for perturbations around the metric of Λ -instantons - not just around global AdS_4 space-time - and find their boundary manifestation by holographic techniques. We hope to return to this problem elsewhere.

Finally, it will be very interesting to explore the role of gravitational duality in the two-point functions of the energy-momentum tensor for spherical perturbations of AdS_4 space-time, and more generally of AdS_4 Schwarzschild solution, using the results presented here as well as those reported in ref. [7]. Work in this direction is in progress, [30].

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